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# ***Resolution of Solvable Equations of the Fifth Degree.***

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§1. By means of the laws established in the paper entitled “Principles of the Solution of Equations of the Higher Degrees,” which is concluded in the present issue of the Journal of Mathematics, a criterion of the solvability of equations of the fifth degree may be found, and the roots of solvable quintics obtained in terms of given numerical coefficients. In certain classes of cases, the roots can be determined in terms of coefficients to which particular numerical values have not been assigned, but which are only assumed to be so related as to make the equations solvable.

## SKETCH OF THE METHOD EMPLOYED.

§2. Let  $r_1, r_2, r_3, r_4, r_5$ , be the roots of the solvable irreducible equation of the fifth degree wanting the second term,

$$F(x) = x^5 + p_2x^3 + p_3x^2 + p_4x + p_5 = 0. \quad (1)$$

It was proved in the "Principles" that

$$r_1 = \frac{1}{5}(\Delta_1^{\frac{1}{5}} + \Delta_2^{\frac{1}{5}} + \Delta_3^{\frac{1}{5}} + \Delta_4^{\frac{1}{5}}),$$

where  $\Delta_1, \Delta_2, \Delta_3, \Delta_4$  are the roots of a biquadratic equation auxiliary to the equation  $F(x) = 0$ . It was also shown that the root can be expressed in the form

$$r_1 = \frac{1}{5}(\Delta_1^{\frac{1}{5}} + a_1\Delta_1^{\frac{2}{5}} + e_1\Delta_1^{\frac{3}{5}} + h_1\Delta_1^{\frac{4}{5}}), \quad (2)$$

where  $a_1, e_1, h_1$ , involve only surds occurring in  $\Delta_1$ ; and no surds occur in  $\Delta_1$  except  $\sqrt{hz + h\sqrt{z}}$  and its subordinate  $\sqrt{z}$ ;  $z$  being equal to  $1 + e^2$ , and  $h$  and  $e$  being rational. As in the "Principles," we may put  $5u_1 = \Delta_1^{\frac{1}{5}}$ ,  $5u_2 = \Delta_2^{\frac{1}{5}}$ ,  $5u_3 = \Delta_3^{\frac{1}{5}}$ ,  $5u_4 = \Delta_4^{\frac{1}{5}}$ . Then

$$r_1 = u_1 + u_2 + u_3 + u_4. \quad (3)$$

Let  $S_1$  be the sum of the roots of the equation  $F(x) = 0$ ,  $S_2$  the sum of their squares, and so on. Also let

$$\left. \begin{aligned} \Sigma(u_1^2 u_3) &= u_1^2 u_3 + u_2^2 u_1 + u_3^2 u_4 + u_4^2 u_2, \\ \Sigma(u_1^3 u_2) &= u_1^3 u_2 + u_2^3 u_4 + u_3^3 u_1 + u_4^3 u_3, \\ \Sigma(u_1 u_2^2 u_4^2) &= u_1 u_2^2 u_4^2 + u_2 u_1^2 u_3^2 + u_3 u_4^2 u_2^2 + u_4 u_2^2 u_1^2, \\ \Sigma(u_1^5) &= u_1^5 + u_2^5 + u_3^5 + u_4^5. \end{aligned} \right\} \quad (4)$$

Then

$$\left. \begin{aligned} S_2 &= 10(u_1 u_4 + u_2 u_3), \quad S_3 = 15 \{\Sigma(u_1^2 u_3)\}, \\ S_4 &= 20 \{\Sigma(u_1^3 u_2)\} + \frac{3}{10}(S_2^2) + 60 u_1 u_2 u_3 u_4, \\ S_5 &= 5 \{\Sigma(u_1^5)\} + \frac{2}{3}(S_2 S_3) + 50 \{\Sigma(u_1 u_2^2 u_4^2)\}. \end{aligned} \right\}$$

§3. It was proved in the "Principles" that  $u_1 u_4$  and  $u_2 u_3$  are the roots of a quadratic equation. But  $25 u_1 u_4 = h_1 \Delta_1$ , and  $25 u_2 u_3 = a_1 e_1 \Delta_1$ . Therefore, because  $a_1, e_1, h_1$ , involve no surds that are not subordinate to  $\Delta_1^{\frac{1}{5}}$ ,  $\sqrt{z}$  is the only surd that can appear in  $u_1 u_4$  and  $u_2 u_3$ . Consequently we may put

$$u_1 u_4 = g + a\sqrt{z}, \text{ and } u_2 u_3 = g - a\sqrt{z}, \quad (5)$$

where  $g, a$ , are rational. It scarcely needs to be pointed out that these forms are valid whether the surd  $\sqrt{z}$  is irreducible or not. Now  $S_2 = 10(u_1 u_4 + u_2 u_3) = -2p_2$ . Therefore

$$g = -\frac{1}{10}(p_2). \quad (6)$$

Again, it was shown in the "Principles" that the four expressions  $u_1^2 u_3, u_2^2 u_1, u_3^2 u_4, u_4^2 u_2$ , are the roots of a biquadratic equation. And, by the same reasoning as that employed in the case of  $u_1 u_4$  and  $u_2 u_3$ , the only surds that can appear in

these expressions are  $\sqrt[3]{hz + h\sqrt[3]{z}}$ ,  $\sqrt[3]{hz - h\sqrt[3]{z}}$ , and  $\sqrt[3]{z}$ . Let  $hz + h\sqrt[3]{z} = s$ , and  $hz - h\sqrt[3]{z} = s_1$ . Then

$$\sqrt[3]{s_1} = \left( \frac{\sqrt[3]{z} - 1}{e} \right) \sqrt[3]{s}, \text{ and } \sqrt[3]{s} \sqrt[3]{s_1} = he \sqrt[3]{z}. \quad (7)$$

Hence the expressions  $u_1^2 u_3$ ,  $u_2^2 u_1$ ,  $u_3^2 u_4$ ,  $u_4^2 u_2$ , may have their value exhibited in terms of  $\sqrt[3]{z}$  and either of the surds  $\sqrt[3]{s}$ ,  $\sqrt[3]{s_1}$ . Put

$$\left. \begin{aligned} u_1^2 u_3 &= k + c\sqrt[3]{z} + (\theta + \phi\sqrt[3]{z})\sqrt[3]{s}, \\ u_4^2 u_2 &= k + c\sqrt[3]{z} - (\theta + \phi\sqrt[3]{z})\sqrt[3]{s}, \\ u_2^2 u_1 &= k - c\sqrt[3]{z} + (\theta - \phi\sqrt[3]{z})\sqrt[3]{s_1}, \\ u_3^2 u_4 &= k - c\sqrt[3]{z} - (\theta - \phi\sqrt[3]{z})\sqrt[3]{s_1}; \end{aligned} \right\} \quad (8)$$

where  $k, c, \theta, \phi$ , are rational. These coefficients must bear a relation to  $g, a$ , in (5). In fact, because  $(u_1^2 u_3)(u_4^2 u_2) = (u_1 u_4)^2 (u_2 u_3)$ ,

$$(g^2 - a^2 z)(g + a\sqrt[3]{z}) = (k + c\sqrt[3]{z})^2 - (\theta + \phi\sqrt[3]{z})^2 (hz + h\sqrt[3]{z}).$$

Equating the rational parts to one another, and also the irrational parts,

$$\left. \begin{aligned} hz(\theta^2 + \phi^2 z + 2\theta\phi) &= k^2 + c^2 z - g(g^2 - a^2 z), \\ h(\theta^2 + \phi^2 z + 2\theta\phi z) &= 2kc - a(g^2 - a^2 z). \end{aligned} \right\} \quad (9)$$

Because  $s_2 = 15\{\Sigma(u_1^2 u_3)\} = -3p_3$ ,

$$k = -\frac{1}{20}(p_3). \quad (10)$$

It will be convenient to retain the symbols  $g$  and  $k$ , whose values are given in (6) and (10). Again, because  $u_1^3 u_2 = \frac{(u_1^2 u_3)(u_2^2 u_1)}{u_2 u_3}$ , we have, from (5) and (8),

$$\begin{aligned} u_1^3 u_2 &= \frac{g + a\sqrt[3]{z}}{g^2 - a^2 z} \{k + c\sqrt[3]{z} + (\theta + \phi\sqrt[3]{z})\sqrt[3]{s}\} \{k - c\sqrt[3]{z} + (\theta - \phi\sqrt[3]{z})\sqrt[3]{s_1}\} \\ &= A + A'\sqrt[3]{z} + (A'' + A'''\sqrt[3]{z})\sqrt[3]{s}, \end{aligned}$$

where  $A, A', A'', A'''$ , are rational. The value of  $A$  is

$$A = \frac{1}{g^2 - a^2 z} \{g(k^2 - c^2 z) + ahcz(\theta^2 - \phi^2 z)\}. \quad (11)$$

Again,  $u_1^5 = \frac{(u_1^2 u_3)^2 (u_2^2 u_1)}{(u_2 u_3)^2}$ . That is, from (8) and (5) and (7),

$$\begin{aligned} u_1^5 &= \frac{(g + a\sqrt[3]{z})^2}{(g^2 - a^2 z)^2} \{2(k + c\sqrt[3]{z})^2 - (g^2 - a^2 z)(g + a\sqrt[3]{z}) + 2(k + c\sqrt[3]{z})(\theta + \phi\sqrt[3]{z})\sqrt[3]{s}\} \\ &\quad \{k - c\sqrt[3]{z} + (\theta - \phi\sqrt[3]{z})\sqrt[3]{s_1}\} = B + B'\sqrt[3]{z} + (B'' + B'''\sqrt[3]{z})\sqrt[3]{s}; \end{aligned}$$

where  $B, B', B'', B'''$ , are rational. Now, by (4),

$$S_4 = 20\{\Sigma(u_1^3 u_2)\} + \frac{3}{10}(S_2^2) + 60u_1 u_2 u_3 u_4.$$

And  $S_4 = 2p_2^2 - 4p_4$ . Also  $\Sigma(u_1^3 u_2) = 4A$ ; and, by (6),  $10g = -p_2$ ; and, by (5),  $u_1 u_2 u_3 u_4 = g^2 - a^2 z$ . Therefore

$$p_4 = -20A + 5g^2 + 15a^2 z. \quad (12)$$

Again,  $S_5 = 5 \{ \Sigma(u_1^5) \} + \frac{2}{3} (S_2 S_3) + 50 \{ \Sigma(u_1 u_3^2 u_4^2) \}.$

And  $\Sigma(u_1^5) = 4B$ ,  $S_2 S_3 = 6p_2 p_3 = 1200gk$ , and

$$\Sigma(u_1 u_3^2 u_4^2) = u_1 u_4 (u_3^2 u_1 + u_3^2 u_4) + u_2 u_3 (u_1^2 u_3 + u_4^2 u_2).$$

Therefore  $S_5 = 20B + 1000gk - 200acz.$

$$\text{But } S_5 - 5p_2 p_3 + 5p_5 = S_5 - 1000gk + 5p_5 = 0.$$

Therefore

$$p_5 = -4B + 40acz. \quad (13)$$

The values of  $p_4$  and  $p_5$  in (12) and (13) make the quintic

$$F(x) = x^5 + p_2 x^3 + p_3 x^2 + (5g^2 + 15a^2 z - 20A)x + (40acz - 4B) = 0. \quad (14)$$

§4. Assuming the coefficients  $p_2$ ,  $p_3$ , etc., in (1), to be known, the coefficients in the equation  $F(x) = 0$  as exhibited in (14) involve six unknown quantities, namely,  $a$ ,  $c$ ,  $\theta$ ,  $\phi$ ,  $e$ ,  $h$ . The list does not include  $z$ ,  $g$ ,  $k$ ; because  $z = 1 + e^2$ ; and  $g$  and  $k$  are known by (6) and (10). To find the six unknown quantities we have six equations, which are here gathered together.

$$\left. \begin{aligned} p_4 &= -20A + 5g^2 + 15a^2 z, \\ p_5 &= -4B + 40acz, \\ B'' &= 1, \\ B''' &= 0, \\ hz(\theta^2 + \phi^2 z + 2\theta\phi) &= k^2 + c^2 z - g(g^2 - a^2 z) \\ h(\theta^2 + \phi^2 z + 2\theta\phi z) &= 2kc - a(g^2 - a^2 z). \end{aligned} \right\}$$

The first two of these equations are the equations (12) and (13). As to the third and fourth, it was proved in the "Principles" that the form of  $u_1^5$  is  $m + n\sqrt{z} + \sqrt{hz + h\sqrt{z}}$ ,  $m$  and  $n$  being rational. This is saying in other words that  $B'' = 1$  and  $B''' = 0$ . The last two of the equations (15) are the equations (9).

§5. The criterion of solvability of the equation  $F(x) = 0$  may now be stated in a general way to be that the coefficients  $p_2$ ,  $p_3$ , etc., must be so related that rational quantities,  $a$ ,  $c$ ,  $\theta$ ,  $\phi$ ,  $e$ ,  $h$ , exist satisfying the equations (15). We also see what requires to be done in order to find the roots of the equation  $F(x) = 0$  in terms of the given coefficients. By (3),  $r_1$  is known when  $u_1$ ,  $u_2$ ,  $u_3$ ,  $u_4$  are known. But,  $B''$  and  $B'''$  being respectively unity and zero,

$$\begin{aligned} u_1^5 &= B + B'\sqrt{z} + \sqrt{s}, & u_2^5 &= B - B'\sqrt{z} + \sqrt{s_1}, \\ u_4^5 &= B + B'\sqrt{z} - \sqrt{s}, & u_3^5 &= B - B'\sqrt{z} - \sqrt{s_1}. \end{aligned}$$

Therefore, to find  $r_1$  we need to find  $B$ ,  $B'$ ,  $z$  and  $h$ ; which is equivalent to saying that we need to find the six unknown quantities  $a$ ,  $c$ ,  $\theta$ ,  $\phi$ ,  $e$ ,  $h$ . Before pointing out how this may be done in the most general case, I will refer to some special forms of soluble quintics.

CASE IN WHICH  $u_1 u_4 = u_2 u_3$ .

§6. A notable class of solvable quintics is that in which  $u_1 u_4 = u_2 u_3$ . It includes, as was proved in the "Principles," all the Gaussian equations of the fifth degree for the reduction of  $x^n - 1 = 0$ ,  $n$  prime. It includes also other equations, of which examples will presently be given. Now, when  $u_1 u_4 = u_2 u_3$ , the root of the quintic can be found in terms of the coefficients  $p_2, p_3$ , etc., even while these coefficients retain their general symbolic forms; in other words, the root can be found in terms of  $p_2, p_3$ , etc., without definite numerical values being assigned to  $p_2, p_3$ , etc. This I proceed to show.

§7. By (5), because  $u_1 u_4 = u_2 u_3$ ,  $\alpha = 0$ . Thus, one of the six unknown quantities is determined, while we have still the six equations (15) to work with. It might be sufficient to say, that, from six equations five unknown rational quantities can be found. I will recur to this idea; but in the meantime the following line of reasoning may be pursued. From (11),  $A = \frac{k^2 - c^2 z}{g}$ . Therefore equation (12) becomes

$$gp_4 = -20(k^2 - c^2 z) + 5g^3. \quad (16)$$

Also, because  $\alpha = 0$ , equations (7) being kept in view,

$$u_1^5 = \frac{1}{g^2} \{ 2(k^2 - c^2 z)(k + c\sqrt{z}) - g^3(k - c\sqrt{z}) + 2(k + c\sqrt{z})(\theta^2 - \phi^2 z)he\sqrt{z} \} \\ + (B'' + B'''\sqrt{z})\sqrt{s}.$$

$$\therefore Bg^2 = k \{ 2(k^2 - c^2 z) - g^3 \} + 2chez(\theta^2 - \phi^2 z)$$

$$\text{and } B'g^2 = c \{ 2(k^2 - c^2 z) + g^3 \} + 2khe(\theta^2 - \phi^2 z);$$

$$\therefore u_1^5 = \frac{1}{g^2} [k \{ 2(k^2 - c^2 z) - g^3 \} + 2chez(\theta^2 - \phi^2 z)] \\ + \frac{\sqrt{z}}{g^2} [c \{ 2(k^2 - c^2 z) + g^3 \} + 2khe(\theta^2 - \phi^2 z)] + \sqrt{s}. \quad (17)$$

Substitute in the second of equations (15) the value of  $B$  that has been obtained.

$$\text{Then } g^2 p_5 = -4k \{ 2(k^2 - c^2 z) - g^3 \} - 8chez(\theta^2 - \phi^2 z). \quad (18)$$

The values of  $B''$  and  $B'''$  are

$$\left. \begin{aligned} B''eg^2 &= \theta \{ M + 2e(k^2 - c^2 z) \} - \phi zN = eg^2, \\ B'''eg^2 &= \theta N - \phi \{ M - 2e(k^2 - c^2 z) \} = 0; \end{aligned} \right\} \quad (19)$$

$$\left. \begin{aligned} \text{where } M &= -2(k^2 + c^2 z) + g^3 + 4kc z, \text{ which may be written } M = 4kc z - P, \\ \text{and } N &= 2(k^2 + c^2 z) - g^3 - 4kc, \text{ which may be written } N = P - 4kc. \end{aligned} \right\} \quad (20)$$

The two equations (19) give us

$$\left. \begin{aligned} \theta \{ M^2 - zN^2 - 4e^2(k^2 - c^2 z)^2 \} &= eg^2 \{ M - 2e(k^2 - c^2 z) \}, \\ \phi \{ M^2 - zN^2 - 4e^2(k^2 - c^2 z)^2 \} &= eg^2 N. \end{aligned} \right\} \quad (21)$$

$$\text{Therefore } \frac{\theta}{\phi} = \frac{M - 2e(k^2 - c^2 z)}{N}.$$

Equating the value of  $\frac{\theta^2 + \varphi^2 z + 2\theta\varphi}{\theta^2 + \varphi^2 z + 2\theta\varphi z}$  obtained from (21) with that derived from the last two of equations (15),

$$\frac{k^2 + c^2 z - g^3}{2kcz} = \frac{\{M - 2e(k^2 - c^2 z)\}^2 + N^2 z + 2N\{M - 2e(k^2 - c^2 z)\}}{\{M - 2e(k^2 - c^2 z)\}^2 + N^2 z + 2Nz\{M - 2e(k^2 - c^2 z)\}}. \quad (22)$$

The coefficients  $p_2, p_3$ , etc., in the equation  $F(x) = 0$ , being given,  $g$  and  $k$  are known by (6) and (10). Therefore, by (16),  $c^2 z$  is known. Then (22) will be found to be a quadratic equation determinative of  $c$ . For, keeping in view the value of  $P$  in (20), (22) may be written

$$\frac{k^2 + c^2 z - g^3}{2kc^2 z} = \frac{\{4(k^2 + c^2 z)^2 + P^2\} - 8kPc - 16k(k^2 - c^2 z)(ce)}{\{4(k^2 - c^2 z)^2 - 16k^2 c^2 z - P^2\}c + 8kc^2 zP - 4(k^2 - c^2 z)P(ce)}.$$

Because  $g, k, c^2 z$  and  $P$  are known, this equation is of the form

$$H(ce) = Kc + L,$$

where  $H, K, L$ , are known. Therefore, since  $c^2 e^2 = c^2 z - c^2$ ,

$$c^2(H^2 + K^2) + 2KLc + (L^2 - H^2 c^2 z) = 0;$$

from which  $c$  is known. Therefore, since  $c^2 z$  is known,  $z$  is known. Therefore  $e$  is known. Therefore, by (21),  $\theta$  and  $\phi$  are known. Therefore, by (18) or either of the equations (9),  $h$  is known. Therefore, by (17),  $u_1^5$  is known. In like manner,  $u_2^5, u_3^5, u_4^5$ , are known. Hence finally, by (3),  $r_1$  is known.

§8. *Example First.* I will now give some numerical verifications of the theory. The Gaussian equation of the fifth degree for the reduction of  $x^{11} - 1 = 0$ , when deprived of its second term, is

$$x^5 - \frac{22}{5}x^3 - \frac{11}{25}x^2 + \frac{11 \times 42}{125}x + \frac{11 \times 89}{3125} = 0.$$

When a root of this equation is expressed as in (1), the value of  $r_1$ , as given by Lagrange, is

$$u_1^5 = \frac{11}{4(5)^5} \{-89 - 25\sqrt{5} + 5(19 - 9\sqrt{5})(-5 - 2\sqrt{5})\};$$

which, reduced to the form that we have adopted, is

$$u_1^5 = \frac{11}{4(5)^5} \left\{ -89 + 25 \times \frac{89}{41} \sqrt{\left(\frac{5 \times 41^2}{89^2}\right)} \right\} + \sqrt{(hz + h\sqrt{z})};$$

where  $h = -\frac{11^2 \times 89^2}{8 \times 41 \times (5)^8}$ ,  $\sqrt{z} = -\frac{41}{89}\sqrt{5}$ , and  $e = -\frac{22}{89}$ . We have to show that this is the result to which the equations of the preceding section lead. The simplest way will be to find  $g, k$  and  $c^2 z$  by means of (6), (10) and (16), and then to take the values of  $e$  and  $\sqrt{z}$  given above, and to substitute them in equation (22). If the theory is sound, the equation ought in this way to be

satisfied. When this equation has been satisfied, it will be unnecessary to pursue the verification farther. Because  $p_2 = -\frac{22}{5}$ , and  $p_3 = -\frac{11}{25}$ ,  $g = \frac{11}{25}$  and  $k = \frac{11}{20 \times 25}$ . From (18), taken in connection with (21),  $che$  must be negative. Therefore

$$c = -\frac{11 \times 89}{4 \times 25 \times 41}, \quad kc = -\frac{89}{80 \times 41} \left(\frac{11}{25}\right)^2, \quad kcz = -\frac{41}{16 \times 89} \left(\frac{11}{25}\right)^2,$$

$$k^2 - c^2 z = -\frac{31}{100} \left(\frac{11}{25}\right)^2,$$

$$M = -\frac{2716}{89 \times 100} \left(\frac{11}{25}\right)^2, \quad N = \frac{1224}{41 \times 100} \left(\frac{11}{25}\right)^2, \quad M - 2e(k^2 - c^2 z) = -\frac{4080}{89 \times 100} \left(\frac{11}{25}\right)^2.$$

These values reduce the equation (22) to the identity

$$\frac{89}{41} = \frac{89}{41} \left\{ \frac{41(4080^2 + 5 \times 1224^2) - 89(2448 \times 4080)}{89(4080^2 + 5 \times 1224^2) - 205(2448 \times 4080)} \right\}.$$

§9. *Example Second.* The example that has been given is one in which the auxiliary biquadratic is irreducible. I will now take an example,

$$x^5 + 10x^3 - 80x^2 + 145x - 480 = 0, \quad (23)$$

in which the auxiliary biquadratic has a sub-auxiliary quadratic. When the root of the equation (23) is put in the form (1),

$u_1 = (1 + \sqrt{2})^{\frac{1}{2}}, u_4 = (1 - \sqrt{2})^{\frac{1}{2}}, u_2 = (1 + \sqrt{2})(1 + \sqrt{2})^{\frac{3}{2}}, u_3 = (1 - \sqrt{2})(1 - \sqrt{2})^{\frac{3}{2}}$ , the product of the roots  $(1 + \sqrt{2})^{\frac{1}{2}}, (1 - \sqrt{2})^{\frac{1}{2}}$ , being  $-1$ . Putting  $\beta$  for 28560, and  $\lambda$  for 28562,

$$g = -1, \quad k = 4, \quad c\sqrt{z} = -3, \quad z = \frac{\lambda^2}{\beta^2}, \quad c = \frac{3\beta}{\lambda}, \quad k^2 + c^2 z = 25, \quad kc = \frac{12\beta}{\lambda}, \quad kcz = \frac{12\lambda}{\beta},$$

$$P = 2(k^2 + c^2 z) - g^3 = 51, \quad M = \frac{48\lambda - 51\beta}{\beta}, \quad N = \frac{51\lambda - 48\beta}{\lambda},$$

$$M - 2e(k^2 - c^2 z) = \frac{48\lambda - 51\beta + 14 \times 338}{\beta}.$$

These values cause (22) to become

$$\frac{13}{12} = \frac{\lambda \{ (48\lambda - 51\beta + 14 \times 338)^2 + (51\lambda - 48\beta)^2 \} + 2\beta(51\lambda - 48\beta)(48\lambda - 51\beta + 14 \times 338)}{\beta \{ (48\lambda - 51\beta + 14 \times 338)^2 + (51\lambda - 48\beta)^2 \} + 2\lambda(51\lambda - 48\beta)(48\lambda - 51\beta + 14 \times 338)}$$

This may be written  $\frac{13}{12} = \frac{H\lambda + 2K\beta}{H\beta + 2K\lambda}$ . In order that this equation may subsist, it is necessary that

$$H(13\beta - 12\lambda) = 2K(12\beta - 13\lambda); \quad \text{or} \quad \frac{H}{2} \left( \frac{\beta - 24}{2} \right) = -\frac{K(\beta + 26)}{2}$$

But  $H = (-80852)^2 + (85782)^2 = 6537045904 + 7358551524 = 13895597428$ ;  
 $-K = (80852)(85782) = 6935646264$ ;  $\frac{\beta - 24}{2} = 14268$ ;  $\frac{\beta + 26}{2} = 14293$ ;  
 and  $6947798714 \times 14268 = 6935646264 \times 14293 = 99131192051352$ .

§10. *Example Third.* I will finally take an example,

$$x^5 + 20x^3 + 20x^2 + 30x + 10 = 0, \quad (24)$$

in which the roots of the auxiliary biquadratic are all rational. By (6) and (10) and (16),  $g = -2$ ,  $k = -1$ ,  $c^2z = 0$ . Therefore the denominator of the expression on the left of (22) is zero, while the numerator is not zero. Therefore the denominator of the expression on the right of (22) is zero. Or,  $-g^6 + 4k^2g^3 - 8ek^4 + 4eg^3k^2 = 0$ . Therefore  $e = -\frac{12}{5}$ . Therefore  $z = \left(\frac{13}{5}\right)^2$ , and  $c = 0$ . Hence  $M = -10$ ,  $N = 10$ ; and, if

$$D = M^2 - zN^2 - 4e^2(k^2 - c^2z)^2,$$

$D = -104e^2$ . Therefore, by (21),  $\theta = -\frac{1}{12}$ ,  $\phi = \frac{25}{12 \times 13}$ ,  $\theta^2 - \phi^2z = -\frac{1}{6}$ .

Therefore, by (9),  $h = \frac{225}{26}$ . Therefore, using the symbols,  $B$ ,  $B'$ , as in §3,

$$B = -\frac{5}{2}, \quad B' = -\frac{45}{26}, \quad s = h(z + \sqrt{z}) = 81, \quad s_1 = h(z - \sqrt{z}) = 36.$$

Therefore  $w_1^5 = -7 + 9 = 2$ ,  $w_4^5 = -7 - 9 = -16$ ,  $w_2^5 = 2 - 6 = -4$ ,  $w_3^5 = 2 + 6 = 8$ . Hence, by (3),

$$r_1 = 2^{\frac{1}{5}} - 2^{\frac{2}{5}} + 2^{\frac{3}{5}} - 2^{\frac{4}{5}};$$

which is the solution of the equation (24).

§11. It was pointed out in §7 that, in the case we are considering, there are six equations and five unknown quantities. All the unknown quantities may be eliminated, and an equation  $p' = 0$  obtained; where  $p'$  is a rational function of the coefficients  $p_2$ ,  $p_3$ , etc. This elimination has been performed, under the direction of the author of the paper, by Mr. Warren Reid of Toronto, with the following result. Putting  $P$ , as in §7, for  $2(k^2 + c^2z) - g^3$ , let

$$A = -2kc^2zg^3\{8(k^2 + c^2z) - 3g^3\},$$

$$B = g^3\{16k^2c^2z + 4(k^2 + c^2z)^2 - 5g^3(k^2 + c^2z) + g^6\},$$

$$D = -4(k^2 - c^2z)\{-g^6 + 3g^3(k^2 + c^2z) - 2(k^2 - c^2z)^2\},$$

$$A_1 = -8kc^2z[32kc^2z(k^2 - c^2z) - P\{p_5g^2 + 8k(k^2 - c^2z) - 4kg^3\}]$$

$$B_1 = \{p_5g^2 + 8k(k^2 - c^2z) - 4kg^3\}[-32k^2c^2z + g^3\{4(k^2 + c^2z) - g^3\}] + 64kc^2zP(k^2 - c^2z),$$

$$D_1 = -16kc^2zg^3\{4(k^2 + c^2z) - g^3\} + 4P(k^2 - c^2z)\{p_5g^2 + 8k(k^2 - c^2z) - 4kg^3\}.$$

Then, since  $10g = -p_2$ , and  $20k = -p_3$ , and  $20c^2z = p_4g - 5g^3 + 20k^2$ , the quantities  $A, B, D, A_1, B_1, D_1$ , are known rational functions of  $p_2, p_3$ , etc. And

$$(B^2 + D^2)(A_1^2 - D_1^2c^2z) - (B_1^2 + D_1^2)(A^2 - D^2c^2z) + 4\{AB(B_1^2 + D_1^2) - A_1B_1(B^2 + D^2)\}\{AB(A_1^2 - D_1^2c^2z) - A_1B_1(A^2 - D^2c^2z)\} = 0. \quad (25)$$

§12. To verify this result, the Gaussian equation in §8 may be used. Here

$$\begin{aligned} A &= -\frac{11^6}{2^5 \times 5^{12}} \left( \frac{11^3 + 11^2 \times 19}{5^6} \right) = -\frac{11^8 \times 3}{2^4 \times 5^{17}} \\ B &= \frac{11^3}{5^6} \left( \frac{11^4}{2^4 \times 5^9} + \frac{3^4 \times 7^2 \times 11^4}{2^4 \times 5^{12}} - \frac{9 \times 35 \times 11^5}{8 \times 5^{12}} + \frac{11^6}{5^{12}} \right) = -\frac{9 \times 11^7}{4 \times 5^{16}} \\ D &= \frac{11^2 \times 31}{5^{18}} \left( -11^6 + \frac{7 \times 27 \times 11^5}{8} - \frac{31^2 \times 11^4}{8} \right) = \frac{3 \times 31 \times 11^6}{4 \times 5^{16}} \\ A_1 &= \frac{11^8}{2^6 \times 5^{18}} (19 + 31) = \frac{11^8}{2^5 \times 5^{16}} \\ B_1 &= \frac{11^7}{2^4 \times 5^{18}} (-5^3 + 44 \times 41 - 19 \times 31) = \frac{11^7 \times 109}{8 \times 5^{17}} \\ D_1 &= -\frac{11^6}{4 \times 5^{12}} \left( \frac{63 \times 11^2}{2 \times 5^6} - \frac{11^3}{5^6} \right) - \frac{11^7 \times 19 \times 31}{8 \times 5^{18}} = -\frac{11^7 \times 26}{5^{17}}. \end{aligned}$$

$$\begin{aligned} \text{Therefore } B^2 + D^2 &= \frac{9 \times 11^{12} \times 41}{8 \times 5^{30}}, \quad B_1^2 + D_1^2 = \frac{11^4 \times 11029}{2^6 \times 5^{33}}, \\ A^2 - D^2c^2z &= -\frac{9 \times 11^{14} \times 89}{2^6 \times 5^{35}}, \quad A_1^2 - D_1^2c^2z = -\frac{11^{16} \times 40139}{2^{10} \times 5^{37}}. \end{aligned}$$

By the substitution of these values, equation (25) becomes

$$\begin{aligned} \frac{11^{56} \times 3^4}{2^{26} \times 5^{186}} \{6265333^2 - 2886277 \times 13600357\} = \\ \frac{11^{56} \times 3^4}{2^{26} \times 5^{186}} \{39254397600889 - 39254397600889\} = 0 \end{aligned}$$

§13. As an additional verification, the equation

$$x^5 + 10x^3 - 80x^2 + 145x - 480 = 0$$

may be taken. Here, by §9,  $g = -1$ ,  $k = 4$ ,  $k^2 - c^2z = 7$ ,  $k^2 + c^2z = 25$ .

Therefore

$$\begin{aligned} A &= 2^3 \times 3^2 \times 7 \times 29, \quad B = -2 \times 5 \times 17 \times 29, \quad D = 2^3 \times 3 \times 7 \times 29, \\ A_1 &= -2^9 \times 3^4 \times 141, \quad B_1 = 2^4 \times 3 \times 17 \times 2393, \quad D_1 = -2^7 \times 3^2 \times 13 \times 19. \\ B^2 + D^2 &= 2^2 \times 29^2 \times 14281, \quad B_1^2 + D_1^2 = 2^8 \times 3^2 \times 5 \times 338016989, \\ A^2 - D^2c^2z &= 0, \quad A_1^2 - D_1^2c^2z = 2^{14} \times 3^6 \times 5 \times 7 \times 17^2 \times 277. \end{aligned}$$

By the substitution of these values, equation (25) becomes

$$\begin{aligned} 2^{18} \times 3^6 \times 5 \times 7 \times 17^2 \times 29^4 \{277 \times 14281^2 \\ + 5^2 \times 7 \times 338016989 - 2^3 \times 3 \times 141 \times 2393 \times 14281\} = 0. \end{aligned}$$

*The Trinomial Quintic*  $x^5 + p_4x + p_5 = 0$ .

§14. In this case, by (6) and (10),  $g = 0$ , and  $k = 0$ . Therefore, by (11),  
 $A = -\frac{he(\theta^2 - \phi^2z)}{a}$ . Therefore, by (12),

$$p_4 = \frac{20he(\theta^2 - \phi^2z)}{a} + 15a^2z. \quad (26)$$

Also, by §3,  $B = \frac{1}{a^2z} \{-a^3z^2c + 2hecz(\theta^2 - \phi^2z)\}$ . Therefore, by (13),

$$p_5 = -\frac{8hec}{a^2}(\theta^2 - \phi^2z) + 44acz. \quad (27)$$

Hence the quintic becomes

$$F(x) = x^5 + \left\{ \frac{20he(\theta^2 - \phi^2z)}{a} + 15a^2z \right\} x + \left\{ -\frac{8hec}{a^2}(\theta^2 - \phi^2z) + 44acz \right\} = 0. \quad (28)$$

The criterion of solvability of a trinomial quintic of the kind under consideration is therefore that the coefficients  $p_4$  and  $p_5$  be related in the manner indicated in the form (28); while at the same time the last four of equations (15), modified by putting  $g = k = 0$ , subsist between the rational quantities  $a, c, e, h, \theta, \phi$ . From these data, the three following equations may be deduced,  $v$  being put for  $\frac{c^2}{a^3}$ :

$$\left. \begin{aligned} 8ev^3 - 4zv^2 + z(3 - 4e)v - z^2 &= 0, \\ \frac{2p_4}{a^3} + \frac{5p_5}{ac} &= 250z, \\ 4v(zv + 4zv - 8v^2) &= \left(-3z + \frac{p_4}{5a^2}\right)\{z + 4v(e - 1) + 8v^2\}. \end{aligned} \right\} \quad (29)$$

The first of these equations is obtained from a comparison of the two equations (9); the second is obtained by putting  $p_4$  and  $p_5$  respectively equal to the values they have in (28); and the third is obtained by putting  $p_4$  equal to the coefficient of the first power of  $x$  in (28).

§15. If any rational values of  $e$  and  $v$  can be found satisfying the first of equations (29), let such values be taken. Then, from the second and third of (29),  $a^2$  and  $ac$  can be found. Therefore  $a$  and  $c$  are known. Therefore, by (21),  $\theta$  and  $\phi$  are known. Therefore, by (9),  $h$  is known. In this way all the elements for the solution of the quintic are obtained.

§16. For example, the three equations (29) are satisfied by the values,

$$e = \frac{1}{2}, \quad z = v = \frac{5}{4}, \quad c = \frac{25}{2}, \quad a = 5, \quad \therefore \theta = 0, \quad \phi = -\frac{4}{75}, \quad h = \frac{45 \times 25^3}{16}.$$

When these values are substituted in (28), the quintic becomes

$$x^5 + \frac{625x}{4} + 3750 = 0.$$

Then the values of  $u_1^5, u_2^5, u_3^5, u_4^5$ , obtained from the expression for  $u_1^5$  in §3, are

$$\begin{aligned} u_1^5 &= \frac{625}{4} \left\{ -1 - \sqrt{\left(\frac{5}{4}\right)} + \frac{3}{\sqrt{5}} \sqrt{\left(\frac{5}{4} + \sqrt{\frac{5}{4}}\right)} \right\}, \\ u_4^5 &= \frac{625}{4} \left\{ -1 - \sqrt{\left(\frac{5}{4}\right)} - \frac{3}{\sqrt{5}} \sqrt{\left(\frac{5}{4} + \sqrt{\frac{5}{4}}\right)} \right\}, \\ u_2^5 &= \frac{625}{4} \left\{ -1 + \sqrt{\left(\frac{5}{4}\right)} - \frac{3}{\sqrt{5}} \sqrt{\left(\frac{5}{4} - \sqrt{\frac{5}{4}}\right)} \right\}, \\ u_3^5 &= \frac{625}{4} \left\{ -1 + \sqrt{\left(\frac{5}{4}\right)} + \frac{3}{\sqrt{5}} \sqrt{\left(\frac{5}{4} - \sqrt{\frac{5}{4}}\right)} \right\}. \end{aligned}$$

Hence,  $r_1 = u_1 + u_2 + u_3 + u_4 = -1.52887 - 2.25035 + 2.48413 - 3.65639 = -4.95148$ .

#### WHEN ANY RELATION IS ASSUMED BETWEEN THE SIX UNKNOWN QUANTITIES.

§17. In the case in which  $u_1 u_4$  was taken equal to  $u_2 u_3$ , a relation was in fact assumed betwixt the six unknown quantities  $a, c, e, h, \theta, \phi$ ; for, as we saw, to put  $u_1 u_4 = u_2 u_3$  is tantamount to putting  $a = 0$ . Hence, as was noticed in §7, we had only five unknown quantities to be found from six equations. Now, when any relation whatever is assumed betwixt the six unknown quantities, the root of the quintic can be found in terms of the given coefficients  $p_2, p_3$ , etc., without any definite numerical values being assigned to the coefficients, because six rational quantities can always be found from seven equations.

#### THE GENERAL CASE.

§18. We have hitherto been dealing with solvable quintics, assumed to be subject to some condition additional to what is involved in their solvability. We have now to consider how the general case is to be dealt with. That is to say, we here make no supposition regarding the equation of the fifth degree  $F(x) = 0$  except that it wants the second term and is solvable algebraically. In this case it is impossible to find the roots in terms of the coefficients  $p_2, p_3$ , etc., while these coefficients retain their general symbolic forms. But the equations in §3 enable us to find the roots when the coefficients receive any definite numerical values that render the equation solvable. For, we have the six equations (15) to determine the six unknown quantities  $a, c, e, h, \theta, \phi$ ; and we

can eliminate five of the unknown quantities, and obtain an equation involving only one unknown quantity. The unknown quantity appearing in this equation has a rational value; but there are known methods of finding the rational roots of any algebraical equation with definite numerical coefficients. Therefore the unknown quantity can be found. In this way all the six unknown quantities  $a, c, e, h, \theta, \phi$ , can be found. Hence the roots of the quintic can be found.

§19. *Note.*—From my friend, Mr. J. C. Glashan, of Ottawa, who read in manuscript the paper on the “*Principles of the Solution of Equations of the Higher Degrees*,” but did not see the present paper on the “*Resolution of Solvable Equations of the Fifth Degree*,” I learn that, setting out from propositions demonstrated in the “*Principles*,” he has arrived at important conclusions in the theory of Quintics, which will be made public without delay; but he has not communicated to me either his method or the results he has obtained.